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A Metric Property of Period Doubling for Nonisosceles Trapezoidal Maps on an Interval

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I. INTRODUCTION

Beyer and Stein [1] proved the following metric property of period doubling for isosceles trapezoidal maps of an interval into itself: given any isosceles trapezoidal map

$$g(x, e) = \begin{cases} x/e & (0 \leq x \leq e), \\ 1 & (e \leq x \leq 2 - e), \\ (2 - x)/e & (2 - e \leq x \leq 2), \end{cases} \quad (1.1)$$

where $0 < e < 1$. With $H(x, e, \lambda) = \lambda g(x, e)$, $1 < \lambda < 2$, and λ_n^e a root of $H_{2^k}(1, e, \lambda) = 1$, where $H_p = H \circ H \circ \cdots \circ H$ (H composed with itself p times), we call $H_{2^k}(1, e, \lambda)$ the harmonic polynomial in $z = \lambda/e$ corresponding to a pattern $R(k)$; see Section II below. Then for $e \leq 0.99$ the sequence $\{\lambda_n^e\}$ is a convergent increasing sequence, and

$$\lim_{n \rightarrow \infty} \frac{\log\{(\lambda_{n+1}^e - \lambda_n^e)/(\lambda_{n+2}^e - \lambda_{n+1}^e)\}}{\log\{(\lambda_n^e - \lambda_{n-1}^e)/(\lambda_{n+1}^e - \lambda_n^e)\}} = 2 \quad (1.2)$$

which implies that $\{\lambda_n^e\}$ is quadratically convergent. This result is based in part on numerical work done on a computer.

Here, we generalize this result from the class of isosceles trapezoidal maps to that of nonisosceles trapezoidal maps, and we give a proof not involving estimates obtained by computer. For $e \leq 0.82$, Beyer and Stein's result becomes a corollary of ours. In addition, we prove our result for all x belonging to the interval $(e_1, 2 - e_2)$, not only for $x = 1$; see Section II below.

Our main result is: for nonisosceles trapezoids, if $0 < e_1 < 1 < 2 - e_2 < 2$, $\lambda_k(x, e_1, r)$ is a root of $f_k(x, e_1, r; z) = 0$, where $f_k(x, e_1, r; z)$ is defined

by (2.4) below, and

$$e_2 < \min \left\{ \max \left\{ 2/(3e_1 + x), 12e_1/(3e_1 + x)^2 \right\}, \max \left\{ \frac{4}{5}, 8(5 - 2x)/25 \right\} \right\} \quad (1.3)$$

With $0 < e_1 < 1$, then $\{\lambda_n(x, e_1, r)\}$ is a convergent monotone increasing sequence and

$$\lim_{n \rightarrow \infty} \frac{\log\{(\lambda_{n+1}(x, e_1, r) - \lambda_n(x, e_1, r))/(\lambda_{n+2}(x, e_1, r) - \lambda_{n+1}(x, e_1, r))\}}{\log\{(\lambda_n(x, e_1, r) - \lambda_{n-1}(x, e_1, r))/(\lambda_{n+1}(x, e_1, r) - \lambda_n(x, e_1, r))\}} = 2, \quad (1.4)$$

which implies $\{\lambda_n(x, e_1, r)\}$ is quadratically convergent. This generalization has the same form as Beyer and Stein's original result [1] for isosceles trapezoids. We prove (1.4) in Section IV below. We also show that the condition (1.3) is equivalent to a set of statements that relate x , e_1 , and e_2 ; see Lemma 3.4 in Section III.

In Section II, we give a number of definitions and derive a general expression for harmonic polynomials $H_{R(k)}(x, e_1, r; z)$, and a recursion formula for the trapezoidal polynomials $f_k(x, e_1, r; z)$. In Section III, we prove four lemmas and some corollaries needed for the proof of our main result.

II. TRAPEZOIDAL POLYNOMIALS AND THEIR ELEMENTARY PROPERTIES

We study trapezoidal maps of the form

$$g(x, e_1, e_2) = \begin{cases} x/e_1 & (0 \leq x \leq e_1), \\ 1 & (e_1 \leq x \leq 2 - e_2), \\ (2 - x)/e_2 & (2 - e_2 \leq x \leq 2), \end{cases} \quad (2.1)$$

where $0 < e_1, e_2 < 1$ and $0 < e_1 + e_2 < 2$. Let $r = e_1/e_2$, $z = \lambda/e_1$, and $H(x, e_1, r; z) = \lambda g(x, e_1, e_2)$, $1 < \lambda < 2$, then

$$H(x, e_1, r; z) = \begin{cases} zx & (0 \leq x \leq e_1), \\ \lambda & (e_1 \leq x \leq 2 - e_2), \\ rz(2 - x) & (2 - e_2 \leq x \leq 2), \end{cases} \quad (2.2)$$

For reference, we exhibit $H(x, e_1, r; z)$ in Fig. 1. We always assume that $x \in (e_1, 2 - e_2)$.

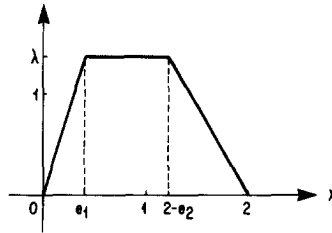


FIGURE 1

Following Metropolis, Stein, and Stein [9], for $x \in (e_1, 2 - e_2)$ we define a pattern P to be the sequence generated by iterating the map $H(x, e_1, r; z)$ $p + 1$ times. The length of P is p . After each application of H an element of P is added to the right. It is an R or an L according as the image of x is to the right of $2 - e_2$ or to the left of e_1 . Next, we denote by $R(k)$ the $(k - 1)$ st harmonic of the pattern R [9]. For example, $R(1) = R$, $R(2) = RLR$, $R(3) = RLR^3LR, \dots$, $R(n + 1) = R(n)PR(n)$, where $P = L$ if $R(n)$ contains an odd number of R 's and $P = R$ otherwise. Given $x \in (e_1, 2 - e_2)$, it can be shown that the pattern $R(n)$ is realizable for some λ [4, 11], so that, for suitable λ , we have

$$H_{R(1)}(x, e_1, r; z) = rz(2 - \lambda) = -e_1 rz^2 + 2rz,$$

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$$H_{R(2)}(x, e_1, r; z) = rz(2 - z(rz(2 - \lambda))) = e_1 r^2 z^4 - 2r^2 z^3 + 2rz,$$

$R \quad L \quad R$

$$H_{R(3)}(x, e_1, r; z) = rz(2 - z(rz(2 - (rz(2 - (rz(2 - z(rz(2 - \lambda) \dots),$$

$R \quad L \quad R \quad R \quad R \quad L \quad R$

$$= -e_1 r^5 z^8 + 2r^5 z^7 - 2r^4 z^5 + 2r^3 z^4 - 2r^2 z^3 + 2rz,$$

$$H_{R(4)}(x, e_1, r; z) = e_1 r^{10} z^{16} - 2r^{10} z^{15} + 2r^9 z^{13} - 2r^8 z^{12} + 2r^7 z^{11}$$

$$- 2r^6 z^9 + 2r^5 z^7 - 2r^4 z^5 + 2r^3 z^4 - 2r^2 z^3 + 2rz, \dots$$

By induction, we see that the general expression for the $(k - 1)$ st harmonic of R is

$$H_{R(k)}(x, e_1, r; z) = (-1)^k e_1 r^{\nu(k)} z^{2^k} + 2 \sum_{n=1}^{\nu(k)} (-1)^{n-1} r^n z^{n+N_L(n)}, \quad (2.3)$$

where $\nu(k)$ is the number of R 's in the pattern $R(k)$ and $N_L(n)$ is the number of L 's before the n th R in $R(k)$, counting from the righthand end

of the pattern $R(k)$. It is easy to verify that $\nu(k) = (2^{k+1} - 2)/3$ if k is even and $\nu(k) = (2^{k+1} - 1)/3$ if k is odd.

Now we may define the trapezoidal polynomial $f_k(x, e_1, r; z)$ in z corresponding to a pattern $R(k)$ as

$$f_k(x, e_1, r; z) \equiv f_{R(k)}(x, e_1, r; z) = H_{R(k)}(x, e_1, r; z) - x. \quad (2.4)$$

Let $\lambda_k(x, e_1, r)$ be a root of the equation $f_{R(k)}(x, e_1, r; z) = 0$ such that the pattern $R(k)$ is realizable for x under the iteration of $H(x, e_1, r; z_k)$, where $z_k = \lambda_k(x, e_1, r)/e_1$; see [4, 8, 11]. Then we observe that

$$\begin{aligned} f_k(x, e_1, r; z) &= -z^{2^{k-1}} r^{\nu_*(k-1)} f_{k-1}(1, e_1, r; z) \\ &\quad + f_{k-1}(x, e_1(k-1), r; z), \end{aligned} \quad (2.5)$$

where $\nu_*(k) = \nu(k)$ if k is odd and $\nu_*(k) = \nu(k) + 1$ if k is even, and $e_1(k) = 1$ if k is odd and $e_1(k) = r$ if k is even.

Next, we take note of two simple properties of f_n that will be used in the sequel.

PROPERTY 2.1.

$$f_n(x, e_1(n), r; z_n) = (-1)^n z_n^{2^n} r^{\nu(n)} (e_1(n) - e_1). \quad (2.6)$$

Proof. By (2.4), (2.3), and since $f_n(x, e_1, r; z_n) = 0$, we have

$$\begin{aligned} f_n(x, e_1(n), r; z_n) &= H_{R(n)}(x, e_1(n), r; z_n) - x, \\ &= (-1)^n e_1(n) r^{\nu(n)} z_n^{2^n} \\ &\quad + 2 \sum_{m=1}^{\nu(n)} (-1)^{m-1} r^m z_n^{m+N_L(m)} - x \\ &\quad + (-1)^n e_1 r^{\nu(n)} z_n^{2^n} - (-1)^n e_1 r^{\nu(n)} z_n^{2^n}, \\ &= H_{R(n)}(x, e_1, r; z_n) - x \\ &\quad + (-1)^n r^{\nu(n)} z_n^{2^n} (e_1(n) - e_1), \\ &= (-1)^n r^{\nu(n)} z_n^{2^n} (e_1(n) - e_1). \end{aligned}$$

PROPERTY 2.2.

$$\begin{aligned} f_n(x, e_1, r; z_{n-1}) &= (-1)^{n-1} r^{\nu(n-1)} z_{n-1}^{2^{n-1}} (e_1(n-1) - e_1) \\ &\quad + (1-x) r^{\nu_*(n-1)} z_{n-1}^{2^{n-1}}. \end{aligned} \quad (2.7)$$

$$f_n(x, e_1, r; z_{n-1}) > 0 \quad (n \text{ odd}), \quad f_n(x, e_1, r; z_{n-1}) < 0 \quad (n \text{ even}).$$

In particular, if $x = 1$, then

$$f_n(1, e_1, r; z_{n-1}) = (-1)^{n-1} r^{\nu(n-1)} z_{n-1}^{2^{n-1}} (e_1(n-1) - e_1).$$

Proof. By (2.5) and Property 2.1, we have

$$\begin{aligned} f_n(x, e_1, r; z_{n-1}) &= -z_{n-1}^{2^{n-1}} r^{\nu(n-1)} f_{n-1}(1, e_1, r; z_{n-1}) \\ &\quad + f_{n-1}(x, e_1(n-1), r; z_{n-1}), \\ &= -z_{n-1}^{2^{n-1}} r^{\nu(n-1)} (H_{R(n-1)}(x, e_1, r; z_{n-1}) - 1 - x + x) \\ &\quad + f_{n-1}(x, e_1(n-1), r; z_{n-1}), \\ &= (1-x) r^{\nu(n-1)} z_{n-1}^{2^{n-1}} \\ &\quad + (-1)^{n-1} r^{\nu(n-1)} z_{n-1}^{2^{n-1}} (e_1(n-1) - e_1). \end{aligned}$$

It is not difficult to verify the second part of Property 2.2, which is used to obtain (4.10) from (4.5), (3.2), (3.5), and (3.6) below.

III. PRELIMINARY LEMMAS

LEMMA 3.1. For arbitrary $e_1, e_2 \in (0, 1)$, $rz_i > 1$ ($i = 1, 2, \dots$).

Proof. We consider rz_1 , where z_1 satisfies the equation $f_1(x, e_1, r; z_1) = 0$, that is, $-e_1 rz_1^2 + 2rz_1 - x = 0$. We get

$$z_1 = [1 + (1 - e_2 x)^{1/2}] / e_1,$$

which is real because $x < 2 - e_2$ and $-(e_2 - 1)^2 < 0$ imply $x < 1/e_2$. Then $rz_1 = [1 + (1 - e_2 x)^{1/2}] / e_2 > 1$. As is known [8, 9], $z_1 \leq z_2 \leq \dots \leq z_n \leq \dots$; a proof is given in [11]. Therefore, $rz_i > 1$ ($i = 1, 2, \dots$).

LEMMA 3.2. Suppose e_1 and e_2 are given with $0 < e_1 + e_2 < 2$, $0 < e_1, e_2 < 1$ and $x \in (e_1, 2 - e_2)$ is fixed. Then $rz_1^2 - 3 > 0$ if and only if

$$e_2 < \max\{2/(3e_1 + x), 12e_1/(3e_1 + x)^2\};$$

and $4rz_1 - 5 > 0$ if and only if $e_2 < \max\{\frac{4}{5}, 8(5 - 2x)/25\}$. In addition, put $t_1 = 2x^2/[3(5 - 2x)]$, $t_2 = (5 - 2x)/6$, $y_1(t) = 2/(3t + x)$, and $y_2(t) = 12t/(3t + x)^2$. Then $y_2(t_i) = 8(5 - 2x)/25$. Moreover, $t_1 > t_2$ and $\frac{4}{5} > 8(5 - 2x)/25$ if $x > \frac{5}{4}$; $t_1 < t_2$ and $\frac{4}{5} < 8(5 - 2x)/25$ if $x < \frac{5}{4}$; and $t_1 = t_2$, $\frac{4}{5} = 8(5 - 2x)/25$ if $x = \frac{5}{4}$.

Proof. $rz_1^2 - 3 > 0$ is equivalent to $-3 + (2rz_1 - x)/e_1 > 0$, i.e.,

$$(1 - e_2 x)^{1/2} > e_2(3e_1 + x)/2 - 1$$

is equivalent to either $e_2 < 2/(3e_1 + x)$ or $e_2 < 12e_1/(3e_1 + x)^2$. Consequently $rz_1^2 - 3 > 0$ if and only if

$$e_2 < \max\{2/(3e_1 + x), 12e_1/(3e_1 + x)^2\}.$$

Similarly, $4rz_1 - 5 > 0$ is equivalent to $(1 - e_2x)^{1/2} > \frac{5}{4}e_2 - 1$, and this in turn is equivalent to either $e_2 < \frac{4}{5}$ or $e_2 < 8(5 - 2x)/25$. Therefore $4rz_1 - 5 > 0$ if and only if $e_2 < \max\{\frac{4}{5}, 8(5 - 2x)/25\}$, which is equivalent to $e_2 < \frac{4}{5}$ if $x \geq \frac{5}{4}$ and to $e_2 < 8(5 - 2x)/25$ if $x < \frac{5}{4}$. The last part of the lemma is trivially verified.

LEMMA 3.3. *There exists a positive upper bound c_1 , independent of n but depending on e_1 and e_2 , such that*

$$|f'_n(x, e_1, r; z_{n-1})| / [r^{\nu(n)} z_{n-1}^{2^n}] \leq c_1(e_1, e_2).$$

Proof. We give the proof for even n , and we indicate the proof for odd n . For the remainder of the proof, n is even unless specifically stated to be odd. Taking the derivative of (2.4). We have

$$\begin{aligned} f'_n(x, e_1, r; z) &= e_1 r^{\nu(n)} 2^n z^{2^n-1} - 2r^{\nu(n)} (2^n - 1) z^{2^n-2} \\ &\quad + 2r^{\nu(n)-1} (2^n - 3) z^{2^n-4} - 2r^{\nu(n)-2} (2^n - 4) z^{2^n-5} \\ &\quad + \dots - 2r^{[\nu(n)/2]+1} [2^n - (2^n - 2^{n-1} - 1)] z^{2^{n-1}} \\ &\quad + 2r^{\nu(n)/2} [2^n - (2^n - 2^{n-1} + 1)] z^{2^{n-1}-2} \\ &\quad - \dots - 2r^2 [2^n - (2^n - 3)] z^2 + 2r [2^n - (2^n - 1)]. \end{aligned} \quad (3.1)$$

If n is odd, then

$$\begin{aligned} f'_n(x, e_1, r; z) &= -e_1 r^{\nu(n)} 2^n z^{2^n-1} + 2r^{\nu(n)} (2^n - 1) z^{2^n-2} \\ &\quad - 2r^{\nu(n)-1} (2^n - 3) z^{2^n-4} + \dots \\ &\quad - 2r^{[\nu(n)+1]/2+1} [2^n - (2^n - 2^{n-1} - 1)] z^{2^{n-1}} \\ &\quad + 2r^{[\nu(n)+1]/2} [2^n - (2^n - 2^{n-1})] z^{2^{n-1}-1} \\ &\quad - \dots - 2r^2 [2^n - (2^n - 3)] z^2 + 2r [2^n - (2^n - 1)]. \end{aligned} \quad (3.2)$$

We write $f'_n(x, e_1, r; z) = g_n^1(x, e_1, r; z) + g_n^2(x, e_1, r; z) + h_n(r; z)$, where

$$\begin{aligned} g_n^1(x, e_1, r; z) &= 2^n \left\{ -z^{2^{n-1}-1} r^{\nu(n)/2} \right. \\ &\quad \times \left[-e_1 r^{\nu(n)/2} z^{2^{n-1}} + 2 r^{\nu(n)/2} z^{2^{n-1}-1} \right. \\ &\quad \left. \left. - 2 r^{[\nu(n)/2]-1} z^{2^{n-1}-3} + \dots + 2rz - x + x \right] \right\}, \\ &= 2^n \left\{ -z^{2^{n-1}-1} r^{\nu(n)/2} [f_{n-1}(x, e_1, r; z) + x] \right\} \quad (3.3) \end{aligned}$$

$$\begin{aligned} g_n^2(x, e_1, r; z) &= 2^n \left\{ z^{-1} \left[e_1 r^{\nu(n)/2} z^{2^{n-1}} - e_1 r^{\nu(n)/2} z^{2^{n-1}} + 2 r^{\nu(n)/2} z^{2^{n-1}-1} \right. \right. \\ &\quad \left. \left. - \dots - 2r^2 z^3 + 2rz - x + x \right] \right\}, \\ &= 2^n \left\{ z^{-1} [e_1 r^{\nu(n)/2} z^{2^{n-1}} + f_{n-1}(x, e_1, r; z) + x] \right\}, \quad (3.4) \end{aligned}$$

$$\begin{aligned} g_n^1(x, e_1, r; z_{n-1}) + g_n^2(x, e_1, r; z_{n-1}) \\ &= 2^n \left\{ -x r^{\nu(n)/2} z_{n-1}^{2^{n-1}-1} + e_1 r^{\nu(n)/2} z_{n-1}^{2^{n-1}-1} + x z_{n-1}^{-1} \right\}, \\ &= 2^n \left\{ r^{\nu(n)/2} z_{n-1}^{2^{n-1}-1} (e_1 - x) + x z_{n-1}^{-1} \right\}. \end{aligned}$$

Since $z_{n-1} > 1$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{g_n^1(x, e_1, r; z_{n-1}) + g_n^2(x, e_1, r; z_{n-1})}{r^{\nu(n)/2} z_{n-1}^{2^{n-1}-1}} &= \lim_{n \rightarrow \infty} \frac{2^n (e_1 - x)}{r^{\nu(n)/2} z_{n-1}^{2^{n-1}-1}} \\ &= 0, \quad (3.5) \end{aligned}$$

This is obvious for $r \geq 1$. If $r < 1$, then $r^{\nu(n)/2} z_{n-1}^{2^{n-1}-1} \geq (r z_{n-1})^{2^{n-1}-1}$ (note that $\nu(n)/2 < 2^{n-1}$). By Lemma 3.1, (3.5) is also true for $r < 1$.

If n is odd, then

$$\begin{aligned} g_n^1(x, e_1, r; z) &= 2^n \left\{ -z^{2^{n-1}-1} r^{[\nu(n)+1]/2} [f_{n-1}(x, e_1, r; z) + x] \right\}, \\ g_n^2(x, e_1, r; z) &= 2^n \left\{ z^{-1} \left[z^{2^{n-1}} r^{[\nu(n)-1]/2} (2r - e_1) \right. \right. \\ &\quad \left. \left. + f_{n-1}(x, e_1, r; z) + x \right] \right\}, \\ g_n^1(x, e_1, r; z_{n-1}) + g_n^2(x, e_1, r; z_{n-1}) \\ &= 2^n \left\{ r^{[\nu(n)-1]/2} z_{n-1}^{2^{n-1}-1} (2r - e_1 - rx) + x z_{n-1}^{-1} \right\}. \end{aligned}$$

It is easy to see that (3.5) is also true for odd n .

From (3.1) and (3.2), we have

$$\begin{aligned}
 h_n(r; z) &= f'_n(x, e_1, r; z) - g_n^1(x, e_1, r; z) - g_n^2(x, e_1, r; z), \\
 &= 2r^{\nu(n)}z^{2^n-2} - 2 \cdot 3r^{\nu(n)-1}z^{2^n-4} \\
 &\quad + 2 \cdot 4r^{\nu(n)-2}z^{2^n-5} - 2 \cdot 5r^{\nu(n)-3}z^{2^n-6} \\
 &\quad + \dots + 2(2^n - 2^{n-1} - 1)r^{[\nu(n)/2]+1}z^{2^{n-1}} \\
 &\quad - 2(2^n - 2^{n-1} + 1)r^{\nu(n)/2}z^{2^{n-1}} - 2 \\
 &\quad + \dots + 2(2^n - 3)r^2z^2 - 2(2^n - 1)r, \\
 &= 2\{(rz^2 - 3)r^{\nu(n)-1}z^{2^n-4} + (4rz - 5)r^{\nu(n)-3}z^{2^n-6} \\
 &\quad + (7rz^2 - 9)r^{\nu(n)-5}z^{2^n-10} + \dots \\
 &\quad + [(2^n - 2^{n-1} - 1)rz^2 - (2^n - 2^{n-1} + 1)] \\
 &\quad \times r^{\nu(n)/2}z^{2^{n-1}-2} + \dots + [(2^n - 3)rz^2 - (2^n - 1)]r\}. \quad (3.6)
 \end{aligned}$$

Note that the general term of $h_n(r; z)$ is of the form

$$2(raz^{b-a} - b)z^{2^n-(b-1)}r^{\nu(n)-\lambda_{b+1}},$$

where $\lambda_{b+1} \leq b + 1$ and $1 \leq \lambda_{b+1} < \nu(n)$, and $b - a = 1$ or 2 . If n is odd, then we have

$$\begin{aligned}
 h_n(r; z) &= 2\{-(rz^2 - 3)r^{\nu(n)-1}z^{2^n-4} - (4rz - 5)r^{\nu(n)-3}z^{2^n-6} \\
 &\quad - \dots - [(2^n - 4)rz - (2^n - 3)]r^2z^2 - (2^n - 1)r\}.
 \end{aligned}$$

Then we can find an upper bound for $K_n \equiv |h_n(r; z_{n-1})/[r^{\nu(n)}z_{n-1}^{2^n}]|$. First, we observe that

$$\begin{aligned}
 K_n &< 2\left\{\sum_{b+1=4}^{2^n} |raz_{n-1}^{b-a} - b|r^{\nu(n)-\lambda_{b+1}}z_{n-1}^{2^n-(b+1)}\right\} / [r^{\nu(n)}z_{n-1}^{2^n}] \\
 &\leq 2\left\{\sum_{b+1=4}^{2^n} (rz_{n-1}^2 + 1)b / [r^{\lambda_{b+1}}z_{n-1}^{b+1}]\right\} \\
 &< 2(4re_1^{-2} + 1) \sum_{b+1=4}^{\infty} b / [r^{\lambda_{b+1}}z_{n-1}^{b+1}] < \infty. \quad (3.7)
 \end{aligned}$$

The inequality (3.7) is obvious for $r \geq 1$ (note that $z_{n-1} > 1$). If $r < 1$, then $r^{\lambda_{b+1}}z_{n-1}^{b+1} \geq (rz_{n-1})^{b+1}$; and, by Lemma 3.1, (3.7) is also true for $r < 1$.

The inequality (3.7) obviously gives an upper bound for K_n , which depends on r and e_1 , but is independent of n . Finally, using (3.5) and (3.2), we complete the proof.

LEMMA 3.4. *If*

$$e_2 < \min\{\max\{y_1(e_1), y_2(e_1)\}, \max\{\frac{4}{5}, 8(5-2x)/25\}\}, \quad (3.8)$$

then there exists a $c_2(e_1, e_2) > 0$ such that

$$c_2(e_1, e_2) \leq |f'_n(x, e_1, r; z_{n-1})| / [r^{\nu(n)} z_{n-1}^{2^n}]$$

for n sufficiently large. Further, (3.8) holds if and only if one of the following three cases holds, where $x \in (e_1, 2 - e_2)$ and $0 < e_1 < 1 < 2 - e_2 < 2$:

(1) $x = \frac{5}{4}$, either $0 < e_1 \leq \frac{5}{12}$ and $0 < e_2 < \frac{4}{5}$, or $\frac{5}{12} \leq e_1 < 1$ and $0 < e_2 < 12e_1/(3e_1 + x)^2$;

(2) $x > \frac{5}{4}$, either $0 < e_1 \leq x/3$ and $0 < e_2 < \min\{\frac{4}{5}, 2/(3e_1 + x)\}$, or $x/3 \leq e_1 < 1$ and $0 < e_2 < 12e_1/(3e_1 + x)^2$;

(3) $x < \frac{5}{4}$, either $0 < e_1 \leq (5-2x)/6$ and $0 < e_2 < 8(5-2x)/25$, or $(5-2x)/6 \leq e_1 < 1$ and $0 < e_2 < 12e_1/(3e_1 + x)^2$.

Proof. By the first half of Lemma 3.2, we see that if (3.8) holds, then the first two terms of the final expression for $h_n(r; z)$ in (3.6) are positive. It is easy to verify that every term in $h_n(r; z)$ is also positive, and hence $h_n(r; z)$ is a series of all positive terms. Thus

$$h_n(r; z) > 2(rz_1^2 - 3)r^{\nu(n)-1}z_1^{2^n-4},$$

the first term of $h_n(r; z_1)$. If we set $c'_2(e_1, e_2) = 2[rz_1^2 - 3]/[r(2/e_1)^4]$, then $K_n > c'_2(e_1, e_2)$. Now put $c_2(e_1, e_2) = c'_2(e_1, e_2)/2$. Then (3.5) implies that for n sufficiently large

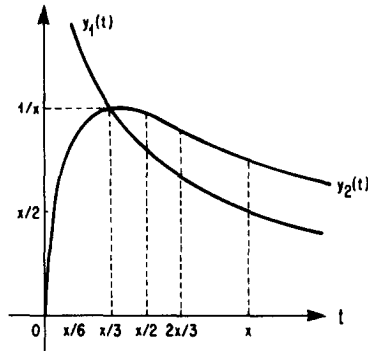
$$\left| \frac{g_n^1(x, e_1, r; z_{n-1}) + g_n^2(x, e_1, r; z_{n-1})}{r^{\nu(n)} z_{n-1}^{2^n}} \right| < c_2(e_1, e_2).$$

Therefore

$$|f'_n(x, e_1, r; z_{n-1})| / [r^{\nu(n)} z_{n-1}^{2^n}] > c_2(e_1, e_2),$$

which is required. Thus we need only to choose e_2 so that (3.8) holds, and the first part of the lemma is proved.

We recall from Lemma 3.2 that $y_1(t) = 2/(3t + x)$ and $y_2(t) = 12t/(3t + x)^2$: see Fig. 2. We now show that the one of the statements (1), (2), and

FIG. 2. $x = 1$.

(3), in the lemma is equivalent to (3.8). First, if $x = \frac{5}{4}$, then, by Lemma 3.2, $\frac{4}{5} = 8(5 - 2x)/25$. Note that $y_2(\frac{5}{12}) = \frac{4}{5}$ and $y_2(t)$ only achieves its maximum $\frac{4}{5}$ at $t = \frac{5}{12}$. Therefore, (3.8) is equivalent to the following: if $0 < e_1 \leq \frac{5}{12}$, then $0 < e_2 < \frac{4}{5}$, and if $\frac{5}{12} \leq e_1 < 1$, then $0 < e_2 < 12e_1/(3e_1 + x)^2$, which is the statement (1) in the lemma.

Next, if $x > \frac{5}{4}$, then, by Lemma 3.2, $\frac{4}{5} > 8(5 - 2x)/25$ and $t_1 > t_2$. The maximum $1/x$ of $y_2(t)$ is less than $\frac{4}{5}$. Thus (3.8) becomes

$$e_2 < \min\left\{\frac{4}{5}, \max\{y_1(e_1), y_2(e_1)\}\right\}.$$

If we choose $e_2 < \min\{\frac{4}{5}, y_1(e_1)\}$ for $e_1 \leq x/3$ and choose $e_2 < y_2(e_1)$ for $x/3 \leq e_1 < 1$, then we obtain the equivalence between (3.8) and the statement (2) of the lemma for $x > \frac{5}{4}$.

If $x < \frac{5}{4}$, then, by Lemma 3.2, $\frac{4}{5} < 8(5 - 2x)/25$ and $t_1 < t_2$. Note that $y_2(t_2) = 8(5 - 2x)/25$. If $e_1 \leq t_2 = (5 - 2x)/6$, then (3.8) becomes $e_2 < 8(5 - 2x)/25$, because

$$\max\{y_1(e_1), y_2(e_1)\} > 8(5 - 2x)/25 = \max\left\{\frac{4}{5}, 8(5 - 2x)/25\right\}.$$

If $t_2 < e_1 < 1$, then (3.8) is equivalent to $e_2 < y_2(e_1)$, because $y_1(e_1) < y_2(e_1) < 8(5 - 2x)/25$. This shows that statement (3) of the lemma is equivalent to (3.8) if $x < \frac{5}{4}$, and the proof of Lemma 3.4 is complete.

If one is interested in the special case $x = 1$, then from statement (3) of Lemma 3.4, one immediately obtains

COROLLARY 3.1. (The case $x = 1$). *There exists $c_2(e_1, e_2) > 0$ such that*

$$c_2(e_1, e_2) \leq |f'_n(x, e_1, r; z_{n-1})| / [r^{\nu(n)} z_{n-1}^{2^n}]$$

for n sufficiently large and either $0 < e_1 \leq \frac{1}{2}$ and $e_2 < \frac{24}{25} = 0.96$, or $\frac{1}{2} \leq e_1 < 1$ and $e_2 < 12e_1/(3e_1 + x)^2$.

Moreover, in the case $x = 1$ and $e_1 = e_2$, then we have, in the notation of Beyer and Stein [1],

COROLLARY 3.2. *If $x = 1$ and $e_1 = e_2 = e \leq 0.82$, then there exist positive numbers $c_1(e)$ and $c_2(e)$ such that*

$$c_2(e) \leq |f'_n(z_{n-1}, e)|/z_{n-1}^{2n} \leq c_1(e)$$

for n sufficiently large.

In the notation of this paper $f'_n(z_{n-1}, e) = f'_n(1, e, 1; z_{n-1})$.

Proof. Put $12e_1/(3e_1 + x)^2 = e_1$. Then $e_1 = \frac{1}{3}(2\sqrt{3} - 1) \approx 0.821367\dots$. If $e_1 \leq 0.82\dots$, then $12e_1/(3e_1 + x)^2 > e_1$ and $\frac{24}{25} > e_1$. Therefore, by Corollary 3.1, we may choose $e_2 = e_1 = e \leq 0.82\dots$. This is the case $e \leq 0.82\dots$ of Lemma 3.3 of Beyer and Stein [1]. (Also see Wang [10].)

IV. THE MAIN RESULT

THEOREM 4.0. *For the class of trapezoidal function studied in Section II, suppose that e_1 , e_2 , and x satisfy one of the three conditions in Lemmas 3.4 and suppose $\lambda_n(x, e_1, r)$ a root of (2.4) with k replaced by n . Then (1.4) holds.*

Proof. The Taylor expansion of $f_n(x, e_1, r; z)$ at z_{n-1} is

$$\begin{aligned} f_n(x, e_1, r; z) &= f_n(x, e_1, r; z_{n-1}) + (z - z_{n-1})f'_n(x, e_1, r; z_{n-1}) \\ &\quad + \frac{1}{2}(z - z_{n-1})^2 f''_n(x, e_1, r; \xi) \quad (z_{n-1} < \xi < z). \end{aligned} \quad (4.1)$$

Since $f_n(x, e_1, r; z_n) = 0$, from (4.1) we obtain the equality

$$\begin{aligned} z_n - z_{n-1} &= -f'_n(x, e_1, r; z_{n-1})/f''_n(x, e_1, r; \xi) \\ &\quad + \frac{\left[f'_n(x, e_1, r; z_{n-1})^2 - 2f''_n(x, e_1, r; \xi)f_n(x, e_1, r; z_{n-1}) \right]^{1/2}}{f''_n(x, e_1, r; \xi)}, \end{aligned} \quad (4.2)$$

where $z_{n-1} < \xi < z_n$. We shall prove that

$$\left| 2f''_n(x, e_1, r; \xi)f_n(x, e_1, r; z_{n-1})/f'_n(x, e_1, r; z_{n-1})^2 \right| < 1, \quad (4.3)$$

for sufficiently large n , so that we can write the Taylor expansion of the expression in the square root in (4.2) as

$$\begin{aligned} & \left[f'_n(x, e_1, r; z_{n-1})^2 - 2f''_n(x, e_1, r; \xi)f_n(x, e_1, r; z_{n-1}) \right]^{1/2} \\ &= f'_n(x, e_1, r; z_{n-1}) \\ & \quad \times \left\{ 1 - f''_n(x, e_1, r; \xi)f_n(x, e_1, r; z_{n-1})/f'_n(x, e_1, r; z_{n-1})^2 \right. \\ & \quad \left. - \frac{1}{2}(1 - \eta)^{-3/2}f''_n(x, e_1, r; \xi)^2f_n(x, e_1, r; z_{n-1})^2 \right. \\ & \quad \left. /f'_n(x, e_1, r; z_{n-1})^4 \right\}, \end{aligned} \quad (4.4)$$

where $0 < |\eta| < 2|f''_n(x, e_1, r; \xi)f_n(x, e_1, r; z_{n-1})|/f'_n(x, e_1, r; z_{n-1})^2$. Thus,

$$\begin{aligned} z_n - z_{n-1} &= -[f_n(x, e_1, r; z_{n-1})/f'_n(x, e_1, r; z_{n-1})] \\ & \quad - \frac{1}{2}(1 - \eta)^{-3/2}f''_n(x, e_1, r; \xi)f_n(x, e_1, r; z_{n-1})^2 \\ & \quad /f'_n(x, e_1, r; z_{n-1})^3. \end{aligned} \quad (4.5)$$

Next, if n is even,

$$\begin{aligned} f''_n(x, e_1, r; z) &= e_1 r^{\nu(n)} 2^n (2^n - 1) z^{2^n - 2} - 2r^{\nu(n)} (2^n - 1)(2^n - 2) z^{2^n - 3} \\ & \quad + 2r^{\nu(n)-1} (2^n - 3)(2^n - 4) z^{2^n - 5} + \dots - 12r^2 z; \end{aligned}$$

hence,

$$|f''_n(x, e_1, r; z)| \leq e_1 r^{\nu(n)} 2^{2n} z^{2^n - 2} \quad (\text{use Lemma 3.1}). \quad (4.6)$$

By Property 2.2, Lemma 3.4, and (4.6), we obtain

$$\begin{aligned} & \left| 2f''_n(x, e_1, r; \xi)f_n(x, e_1, r; z_{n-1})/f'_n(x, e_1, r; z_{n-1})^2 \right| \\ & \leq \left| 2e_1 r^{\nu(n)} 2^{2n} \xi^{2^n - 2} z_{n-1}^{2^n - 1} \right. \\ & \quad \times \left[(-1)^{n-1} r^{\nu(n-1)} (e_1(n-1) - e_1) + (1-x)r^{\nu_*(n-1)} \right] \left| \right. \\ & \quad \left. / \left[c_2(e_1, e_2)^2 z_{n-1}^{2^{n+1}} r^{2\nu(n)} \right] \right| \\ & \leq K 2^{2n} \xi^{2^n} / \left[r^{\nu(n-1)} z_{n-1}^{3 \cdot 2^{n-1}} \right] \\ & = K 2^{2n} / r^{\nu(n-1)} z_{n-1}^{3 \cdot 2^{n-1} - (2^n - 2) \ln \xi / \ln z_{n-1}} \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (4.7)$$

where K is some constant. The result (4.7) is obvious if $r \geq 1$. If $r < 1$, (4.7) also holds because $rz_{n-1} > 1$ and $\nu(n-1) < 2^{n-1}$. Thus for sufficiently large n , (4.3) holds and $|\eta|$ can be made as small as desired.

Now, we estimate the two terms on the right-hand side of (4.5). First, just as in (4.7),

$$\begin{aligned} & \left| f_n(x, e_1, r; z_{n-1})^2 f_n''(x, e_1, r; \xi) / f_n'(x, e_1, r; z_{n-1})^3 \right| \\ & \leq M 2^{2n} / r^{\nu(n)} z_{n-1}^{2 \cdot 2^n - (2^n - 2) \ln \xi / \ln z_{n-1}} \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (4.8)$$

where M is some positive constant. Second, by Property 2.2 and Lemmas 3.3 and 3.4, there exist positive constants N_1 and N_2 such that

$$\begin{aligned} N_1/D < |f_n(x, e_1, r; z_{n-1})/f_n'(x, e_1, r; z_{n-1})| < N_2/D \\ (D = r^{\nu(n-1)} z_{n-1}^{2^{n-1}}), \end{aligned} \quad (4.9)$$

for sufficiently large n .

It now follows from (4.5), (4.8), and (4.9) that there exist positive constants $c_3(e_1, e_2)$ and $c_4(e_1, e_2)$ such that

$$c_3(e_1, e_2)/D < z_n - z_{n-1} < c_4(e_1, e_2)/D \quad \text{or} \quad z_n - z_{n-1} = W_n/D, \quad (4.10)$$

with $c_3(e_1, e_2) < W_n < c_4(e_1, e_2)$, for sufficiently large n . Finally, denoting $\lambda_j(x, e_1, r)$ by λ_j , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\log\{(\lambda_{n+1} - \lambda_n)/(\lambda_{n+2} - \lambda_{n+1})\}}{\log\{(\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n)\}} \\ & = \lim_{n \rightarrow \infty} \frac{\log\{(z_{n+1} - z_n)/(z_{n+2} - z_{n+1})\}}{\log\{(z_n - z_{n-1})/(z_{n+1} - z_n)\}}, \\ & = \lim_{n \rightarrow \infty} \frac{\log(W_{n+1}/W_{n+2}) + 2^n(2 \log z_{n+1} - \log z_n) + (\log r^{\nu(n+1)} - \log r^{\nu(n)})}{\log(W_n/W_{n+1}) + 2^{n-1}(2 \log z_n - \log z_{n-1}) + (\log r^{\nu(n)} - \log r^{\nu(n-1)})}, \\ & = \lim_{n \rightarrow \infty} \frac{2^n(2 \log z_{n+1} - \log z_n) + \frac{1}{3}(2^{n+1} \pm 1) \log r}{2^{n-1}(2 \log z_n - \log z_{n-1}) + \frac{1}{3}(2^n \pm 1) \log r} \\ & \quad (W_{n+1}/W_{n+2} \text{ is bounded as } n \rightarrow \infty), \\ & = \lim_{n \rightarrow \infty} \frac{2^n[(2 \log z_{n+1} - \log z_n) + \frac{2}{3} \log r]}{2^{n-1}[(2 \log z_n - \log z_{n-1}) + \frac{2}{3} \log r]} \\ & \quad (\pm \frac{1}{3} \log r \text{ are constants}) \\ & = 2. \end{aligned}$$

We have finished the proof of Theorem 4.0, i.e. (1.4). From Corollary 3.2 and Theorem 4.0 we immediately have

COROLLARY 4.1. *Let $e_1 = e_2 = e \leq 0.82 \dots$, then (1.4) holds for $x = 1$. (This is Beyer and Stein's theorem for $e \leq 0.82 \dots$ [1].)*

Remark. Beyer and Ebanks [3] have indicated that the sequence $\{(\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n)\}$ diverges for unimodal functions which are constant in a neighborhood of $x = 1$. That means Feigenbaum's conjecture [6] fails for such maps. In fact, from (4.10), it follows that

$$\begin{aligned} (\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n) &= (z_n - z_{n-1})/(z_{n+1} - z_n) \\ &= (W_n/W_{n+1})(z_n/z_{n-1})^{2^{n-1}} z_n^{2^{n-1}} r^{(2^n \pm 1)/3} \rightarrow \infty \\ &\quad \text{as } n \rightarrow \infty. \end{aligned}$$

Beyer and Ebanks also claim that $\{(\lambda_{n+1} - \lambda_n)/(\lambda_n - \lambda_{n-1})^2\}$ converges. But we do not see how to prove this. However, since W_{n+1}/W_n^2 and $(z_{n-1}/z_n)^{2^n}$ are both bounded as $n \rightarrow \infty$, $\lambda_{n+1} - \lambda_n < C(\lambda_n - \lambda_{n-1})^2$ for some $C > 0$; i.e., the sequence $\{\lambda_n(x, e_1, r)\}$ is quadratically convergent.

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